16 Line Integrals

We now need to move on to a new kind of integral. When doing single variable definite integrals we integrated a function of one variable over an interval. In the last chapter we integrated a function of two variables over a two dimensional region and we integrated a function of three variables over a three dimensional solid. In this chapter we are going to look at Line Integrals. The difference in this chapter versus the last chapter is where the values of the variables will come from. For a line integral of a function of two variables the variables will all be on the graph of a two dimensional curve C. Similarly, for a line integral of a function of three variables, the variables will all be on the graph of a three dimensional curve C.

The other main difference in this chapter versus previous chapters in which we evaluated integrals is that in addition to evaluating line integrals over functions we will also, for the first time, be integrating a vector field (which we'll also define).

Once we have a grasp on line integrals and how to compute them we'll take a look at the Fundamental Theorem of Calculus for Line Integrals and it's relationship with conservative vector fields. We will, in addition, discuss a method for determining if a two dimensional vector field is conservative or not and if it is conservative how to find the potential function for the vector field.

Finally, we'll discuss a very important theorem, Green's Theorem. Green's theorem gives a very important relationship between certain line integrals and double integrals.

16.1 Vector Fields

We need to start this chapter off with the definition of a vector field as they will be a major component of both this chapter and the next. Let's start off with the formal definition of a vector field.

Definition

A vector field on two (or three) dimensional space is a function $\vec{F}$ that assigns to each point (x, y) (or (x, y, z)) a two (or three dimensional) vector given by $\vec{F}(x, y)$ (or $\vec{F}(x, y, z)$).

That may not make a lot of sense, but most people do know what a vector field is, or at least they've seen a sketch of a vector field. If you've seen a current sketch giving the direction and magnitude of a flow of a fluid or the direction and magnitude of the winds then you've seen a sketch of a vector field.

The standard notation for the function $\vec{F}$ is,

$$\vec{F}(x, y) = P(x, y)\,\vec{i} + Q(x, y)\,\vec{j}$$
$$\vec{F}(x, y, z) = P(x, y, z)\,\vec{i} + Q(x, y, z)\,\vec{j} + R(x, y, z)\,\vec{k}$$

depending on whether or not we're in two or three dimensions. The function P, Q, R (if it is present) are sometimes called **scalar functions**.

Let's take a quick look at a couple of examples.

Example 1

Sketch each of the following vector fields.

(a) $\vec{F}(x, y) = -y\,\vec{i} + x\,\vec{j}$

(b) $\vec{F}(x, y, z) = 2x\,\vec{i} - 2y\,\vec{j} - 2x\,\vec{k}$

Solution

(a) $\vec{F}(x, y) = -y\,\vec{i} + x\,\vec{j}$

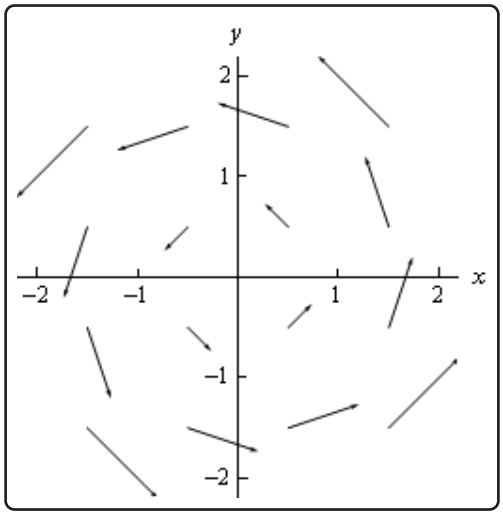
Okay, to graph the vector field we need to get some "values" of the function. This means plugging in some points into the function. Here are a couple of evaluations.

$$\vec{F}\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} \qquad \vec{F}\left(\frac{3}{2}, \frac{1}{4}\right) = -\frac{1}{4}\vec{i} + \frac{3}{2}\vec{j}$$

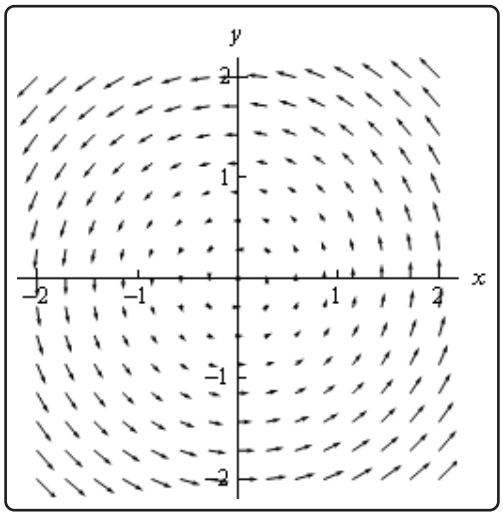
$$\vec{F}\left(\frac{1}{2}, -\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)\vec{i} + \frac{1}{2}\vec{j} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j}$$

So, just what do these evaluations tell us? Well the first one tells us that at the point $\left(\frac{1}{2},\frac{1}{2}\right)$ we will plot the vector $-\frac{1}{2}\vec{i}+\frac{1}{2}\vec{j}$. Likewise, the third evaluation tells us that at the point $\left(\frac{3}{2},\frac{1}{4}\right)$ we will plot the vector $-\frac{1}{4}\vec{i}+\frac{3}{2}\vec{j}$.

We can continue in this fashion plotting vectors for several points and we'll get the following sketch of the vector field.



If we want significantly more points plotted, then it is usually best to use a computer aided graphing system such as Maple or Mathematica. Here is a sketch with many more vectors included that was generated with Mathematica.



(b) $\vec{F}\left(x,y,z\right)=2x\,\vec{i}-2y\,\vec{j}-2x\,\vec{k}$

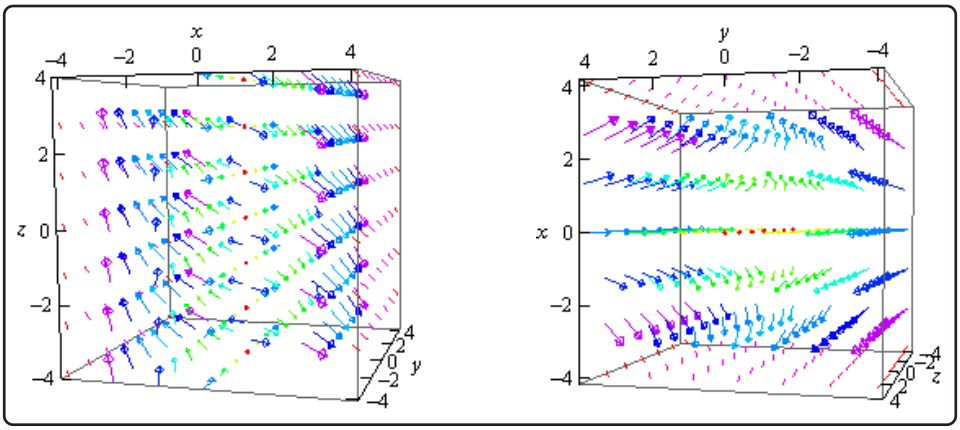
In the case of three dimensional vector fields it is almost always better to use Maple, Mathematica, or some other such tool. Despite that let's go ahead and do a couple of

evaluations anyway.

$$\vec{F}(1,-3,2) = 2\vec{i} + 6\vec{j} - 2\vec{k}$$
$$\vec{F}(0,5,3) = -10\vec{j}$$

Notice that z only affects the placement of the vector in this case and does not affect the direction or the magnitude of the vector. Sometimes this will happen so don't get excited about it when it does.

Here is a couple of sketches generated by Mathematica. The sketch on the left is from the "front" and the sketch on the right is from "above".



Now that we've seen a couple of vector fields let's notice that we've already seen a vector field function. In the second chapter we looked at the gradient vector. Recall that given a function $f(x,y,z)$ the gradient vector is defined by,

Fact

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

This is a vector field and is often called a **gradient vector field**.

In these cases, the function $f(x,y,z)$ is often called a scalar function to differentiate it from the vector field.

Example 2

Find the gradient vector field of the following functions.

(a) $f(x,y) = x^2 \sin(5y)$

(b) $f(x,y,z) = z\mathbf{e}^{-xy}$

Solution

(a) $f(x,y) = x^2 \sin(5y)$

Note that we only gave the gradient vector definition for a three dimensional function, but don't forget that there is also a two dimension definition. All that we need to drop off the third component of the vector.

Here is the gradient vector field for this function.

$$\nabla f = \left\langle 2x \sin(5y), 5x^2 \cos(5y) \right\rangle$$

(b) $f(x,y,z) = z\mathbf{e}^{-xy}$

There isn't much to do here other than take the gradient.

$$\nabla f = \left\langle -yz\mathbf{e}^{-xy}, -xz\mathbf{e}^{-xy}, \mathbf{e}^{-xy} \right\rangle$$

Let's do another example that will illustrate the relationship between the gradient vector field of a function and its contours.

Example 3

Sketch the gradient vector field for $f(x,y) = x^2 + y^2$ as well as several contours for this function.

Solution

Recall that the contours for a function are nothing more than curves defined by,

$$f(x,y) = k$$

for various values of k. So, for our function the contours are defined by the equation,

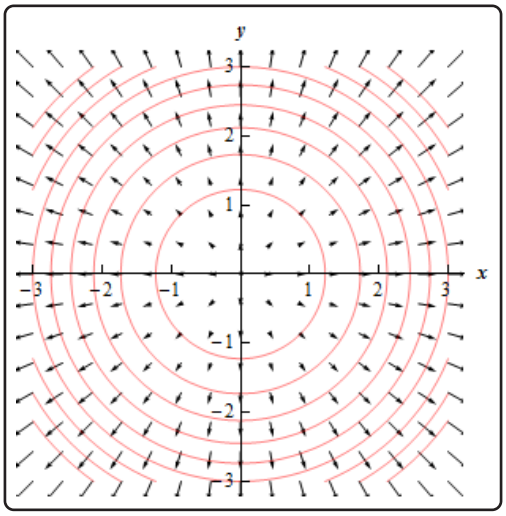
$$x^2 + y^2 = k$$

and so they are circles centered at the origin with radius $\sqrt{k}$.

Here is the gradient vector field for this function.

$$\nabla f\left(x,y\right) = 2x\,\vec{i} + 2y\,\vec{j}$$

Here is a sketch of several of the contours as well as the gradient vector field.



Notice that the vectors of the vector field are all orthogonal (or perpendicular) to the contours. This will always be the case when we are dealing with the contours of a function as well as its gradient vector field.

The k's we used for the graph above were 1.5, 3, 4.5, 6, 7.5, 9, 10.5, 12, and 13.5. Now notice that as we increased k by 1.5 the contour curves get closer together and that as the contour curves get closer together the larger the vectors become. In other words, the closer the contour curves are (as k is increased by a fixed amount) the faster the function is changing at that point. Also recall that the direction of fastest change for a function is given by the gradient vector at that point. Therefore, it should make sense that the two ideas should match up as they do here.

The final topic of this section is that of conservative vector fields. A vector field $\vec{F}$ is called a **conservative vector field** if there exists a function f such that $\vec{F} = \nabla f$. If $\vec{F}$ is a conservative vector field then the function, f, is called a **potential function** for $\vec{F}$.

All this definition is saying is that a vector field is conservative if it is also a gradient vector field for some function.

For instance the vector field $\vec{F} = y\,\vec{i} + x\,\vec{j}$ is a conservative vector field with a potential function of $f\left(x,y\right) = xy$ because $\nabla f = \langle y, x\rangle$.

On the other hand, $\vec{F} = -y\,\vec{i} + x\,\vec{j}$ is not a conservative vector field since there is no function f such that $\vec{F} = \nabla f$. If you're not sure that you believe this at this point be patient, we will be able to prove this in a couple of sections. In that section we will also show how to find the potential function for a conservative vector field.

16.2 Line Integrals - Part I

In this section we are now going to introduce a new kind of integral. However, before we do that it is important to note that you will need to remember how to parameterize equations, or put another way, you will need to be able to write down a set of parametric equations for a given curve. You should have seen some of this in your Calculus II course. If you need some review you should go back and review some of the basics of parametric equations and curves.

Here are some of the more basic curves that we'll need to know how to do as well as limits on the parameter if they are required.

Curve	Parametric Equations
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (Ellipse)	Counter-Clockwise: $x=a\cos(t)$, $y=b\sin(t)$, $0\le t\le 2\pi$ Clockwise: $x=a\cos(t)$, $y=-b\sin(t)$, $0\le t\le 2\pi$
$x^2+y^2=r^2$ (Circle)	Counter-Clockwise: $x=r\cos(t)$, $y=r\sin(t)$, $0\le t\le 2\pi$ Clockwise: $x=r\cos(t)$, $y=-r\sin(t)$, $0\le t\le 2\pi$
$y=f(x)$	$x=t$, $y=f(t)$
$x=g(y)$	$x=g(t)$, $y=t$
Line Segment From (x_0,y_0,z_0) to (x_1,y_1,z_1)	$\vec{r}(t)=(1-t)\langle x_0,y_0,z_0\rangle+t\langle x_1,y_1,z_1\rangle \quad , \quad 0\le t\le 1$ or $x=(1-t)x_0+tx_1$, $y=(1-t)y_0+ty_1$, $z=(1-t)z_0+tz_1$ $\quad , \quad 0\le t\le 1$

With the final one we gave both the vector form of the equation as well as the parametric form and if we need the two-dimensional version then we just drop the z components. In fact, we will be using the two-dimensional version of this in this section.

For the ellipse and the circle we've given two parameterizations, one tracing out the curve clockwise

and the other counter-clockwise. As we'll eventually see the direction that the curve is traced out can, on occasion, change the answer. Also, both of these "start" on the positive x-axis at $t = 0$.

Now let's move on to line integrals. In Calculus I we integrated $f(x)$, a function of a single variable, over an interval $[a, b]$. In this case we were thinking of x as taking all the values in this interval starting at a and ending at b. With line integrals we will start with integrating the function $f(x, y)$, a function of two variables, and the values of x and y that we're going to use will be the points, (x, y), that lie on a curve C. Note that this is different from the double integrals that we were working with in the previous chapter where the points came out of some two-dimensional region.

Let's start with the curve C that the points come from. We will assume that the curve is *smooth* (defined shortly) and is given by the parametric equations,

$$x = h(t) \qquad y = g(t) \qquad a \le t \le b$$

We will often want to write the parameterization of the curve as a vector function. In this case the curve is given by,

$$\vec{r}(t) = h(t)\,\vec{i} + g(t)\,\vec{j} \qquad a \le t \le b$$

The curve is called **smooth** if $\vec{r}\,'(t)$ is continuous and $\vec{r}\,'(t) \ne 0$ for all t.

Defintion

The **line integral** of $f(x, y)$ along C is denoted by,

$$\int_C f(x, y)\, ds$$

We use a ds here to acknowledge the fact that we are moving along the curve, C, instead of the x-axis (denoted by dx) or the y-axis (denoted by dy). Because of the ds this is sometimes called the **line integral of f with respect to arc length**.

We've seen the notation ds before. If you recall from Calculus II when we looked at the arc length of a curve given by parametric equations we found it to be,

$$L = \int_a^b ds\,, \qquad \text{where } ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

It is no coincidence that we use ds for both of these problems. The ds is the same for both the arc length integral and the notation for the line integral.

So, to compute a line integral we will convert everything over to the parametric equations. The line integral is then,

Fact

$$\int_C f(x,y)\,ds = \int_a^b f(h(t),g(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Don't forget to plug the parametric equations into the function as well.

If we use the vector form of the parameterization we can simplify the notation up somewhat by noticing that,

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \left\|\vec{r}\,'(t)\right\|$$

where $\left\|\vec{r}\,'(t)\right\|$ is the magnitude or norm of $\vec{r}\,'(t)$. Using this notation, the line integral becomes,

Fact

$$\int_C f(x,y)\,ds = \int_a^b f(h(t),g(t))\,\left\|\vec{r}\,'(t)\right\|\,dt$$

Note that as long as the parameterization of the curve C is traced out exactly once as t increases from a to b the value of the line integral will be independent of the parameterization of the curve.

Let's take a look at an example of a line integral.

Example 1

Evaluate $\int_C xy^4\,ds$ where C is the right half of the circle, $x^2+y^2=16$ traced out in a counter clockwise direction.

Solution

We first need a parameterization of the circle. This is given by,

$$x = 4\cos(t) \qquad y = 4\sin(t)$$

We now need a range of t's that will give the right half of the circle. The following range of t's will do this.

$$-\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

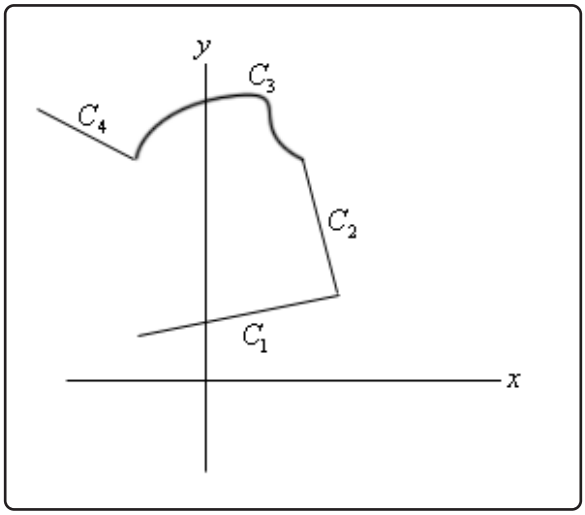
Now, we need the derivatives of the parametric equations and let's compute ds.

$$\frac{dx}{dt} = -4\sin(t) \qquad \frac{dy}{dt} = 4\cos(t)$$
$$ds = \sqrt{16\sin^2(t) + 16\cos^2(t)}\,dt = 4\,dt$$

The line integral is then,

$$\begin{aligned}\int_C xy^4\,ds &= \int_{-\pi/2}^{\pi/2} 4\cos(t)(4\sin(t))^4\,(4)\,dt\\ &= 4096\int_{-\pi/2}^{\pi/2}\cos(t)\ \sin^4(t)\,dt\\ &= \frac{4096}{5}\sin^5(t)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\ &= \frac{8192}{5}\end{aligned}$$

Next we need to talk about line integrals over **piecewise smooth curves**. A piecewise smooth curve is any curve that can be written as the union of a finite number of smooth curves, C_1,..., C_n where the end point of C_i is the starting point of C_{i+1}. Below is an illustration of a piecewise smooth curve.



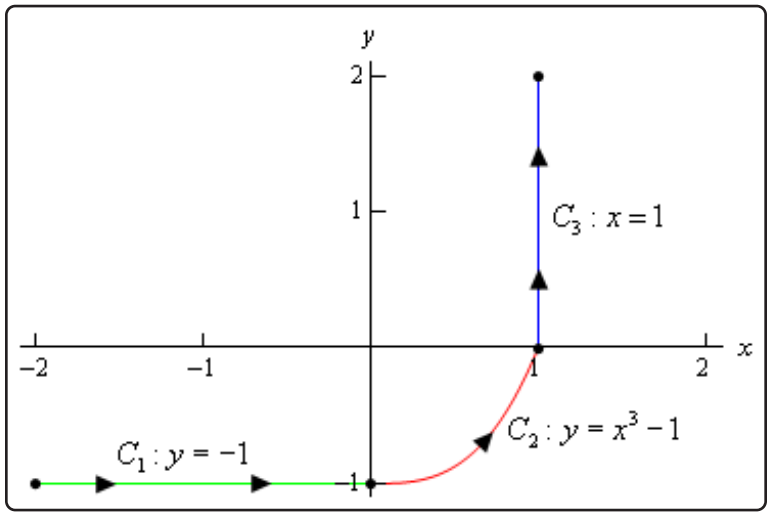
Evaluation of line integrals over piecewise smooth curves is a relatively simple thing to do. All we do is evaluate the line integral over each of the pieces and then add them up. The line integral for some function over the above piecewise curve would be,

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \int_{C_3} f(x,y)\,ds + \int_{C_4} f(x,y)\,ds$$

Let's see an example of this.

Example 2

Evaluate $\int_C 4x^3\,ds$ where C is the curve shown below.



Solution

So, first we need to parameterize each of the curves.

$$\begin{aligned} C_1 &: \quad x = t,\ y = -1\,, \qquad -2 \le t \le 0 \\ C_2 &: \quad x = t,\ y = t^3 - 1, \qquad 0 \le t \le 1 \\ C_3 &: \quad x = 1,\ \ y = t, \qquad\ \ 0 \le t \le 2 \end{aligned}$$

Now let's do the line integral over each of these curves.

$$\int_{C_1} 4x^3\,ds = \int_{-2}^{0} 4t^3\sqrt{(1)^2 + (0)^2}\,dt = \int_{-2}^{0} 4t^3\,dt = t^4\Big|_{-2}^{0} = -16$$

$$\begin{aligned}\int_{C_2} 4x^3\,ds &= \int_0^1 4t^3\sqrt{(1)^2+(3t^2)^2}\,dt\\ &= \int_0^1 4t^3\sqrt{1+9t^4}\,dt\\ &= \frac{1}{9}\left(\frac{2}{3}\right)\left(1+9t^4\right)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{27}\left(10^{\frac{3}{2}}-1\right) = 2.268\end{aligned}$$

$$\int_{C_3} 4x^3\,ds = \int_0^2 4(1)^3\sqrt{(0)^2+(1)^2}\,dt = \int_0^2 4\,dt = 8$$

Finally, the line integral that we were asked to compute is,

$$\begin{aligned}\int_C 4x^3\,ds &= \int_{C_1} 4x^3\,ds + \int_{C_2} 4x^3\,ds + \int_{C_3} 4x^3\,ds\\ &= -16 + 2.268 + 8\\ &= -5.732\end{aligned}$$

Notice that we put direction arrows on the curve in the above example. The direction of motion along a curve *may* change the value of the line integral as we will see in the next section. Also note that the curve can be thought of a curve that takes us from the point $(-2,-1)$ to the point $(1,2)$. Let's first see what happens to the line integral if we change the path between these two points.

Example 3

Evaluate $\int_C 4x^3\,ds$ where C is the line segment from $(-2,-1)$ to $(1,2)$.

Solution

From the parameterization formulas at the start of this section we know that the line segment starting at $(-2,-1)$ and ending at $(1,2)$ is given by,

$$\begin{aligned}\vec{r}\,(t) &= (1-t)\,\langle -2,-1\rangle + t\,\langle 1,2\rangle\\ &= \langle -2+3t, -1+3t\rangle\end{aligned}$$

for $0 \le t \le 1$. This means that the individual parametric equations are,

$$x = -2+3t \qquad y = -1+3t$$

Using this path the line integral is,

$$\begin{aligned}\int_C 4x^3\,ds &= \int_0^1 4(-2+3t)^3\sqrt{9+9}\,dt\\ &= 12\sqrt{2}\left(\frac{1}{12}\right)(-2+3t)^4\Big|_0^1\\ &= 12\sqrt{2}\left(-\frac{5}{4}\right)\\ &= -15\sqrt{2} = -21.213\end{aligned}$$

When doing these integrals don't forget simple Calc I substitutions to avoid having to do things like cubing out a term. Cubing it out is not that difficult, but it is more work than a simple substitution.

So, the previous two examples seem to suggest that if we change the path between two points then the value of the line integral (with respect to arc length) will change. While this will happen fairly regularly we can't assume that it will always happen. In a later section we will investigate this idea in more detail.

Next, let's see what happens if we change the direction of a path.

Example 4

Evaluate $\int_C 4x^3\,ds$ where C is the line segment from $(1,2)$ to $(-2,-1)$.

Solution

This one isn't much different, work wise, from the previous example. Here is the parameterization of the curve.

$$\begin{aligned}\vec{r}(t) &= (1-t)\langle 1,2\rangle + t\langle -2,-1\rangle\\ &= \langle 1-3t, 2-3t\rangle\end{aligned}$$

for $0 \le t \le 1$. Remember that we are switching the direction of the curve and this will also change the parameterization so we can make sure that we start/end at the proper point.

Here is the line integral.

$$\begin{aligned}\int_C 4x^3\,ds &= \int_0^1 4(1-3t)^3\sqrt{9+9}\,dt\\ &= 12\sqrt{2}\left(-\frac{1}{12}\right)(1-3t)^4\Big|_0^1\\ &= 12\sqrt{2}\left(-\frac{5}{4}\right)\\ &= -15\sqrt{2} = -21.213\end{aligned}$$

So, it looks like when we switch the direction of the curve the line integral (with respect to arc length) will not change. This will always be true for these kinds of line integrals. However, there are other kinds of line integrals in which this won't be the case. We will see more examples of this in the next couple of sections so don't get it into your head that changing the direction will never change the value of the line integral.

Before working another example let's formalize this idea up somewhat. Let's suppose that the curve C has the parameterization $x = h(t)$, $y = g(t)$. Let's also suppose that the initial point on the curve is A and the final point on the curve is B. The parameterization $x = h(t)$, $y = g(t)$ will then determine an **orientation** for the curve where the positive direction is the direction that is traced out as t increases. Finally, let $-C$ be the curve with the same points as C, however in this case the curve has B as the initial point and A as the final point, again t is increasing as we traverse this curve. In other words, given a curve C, the curve $-C$ is the same curve as C except the direction has been reversed.

We then have the following fact about line integrals with respect to arc length.

Fact

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$$

So, for a line integral with respect to arc length we can change the direction of the curve and not change the value of the integral. This is a useful fact to remember as some line integrals will be easier in one direction than the other.

Now, let's work another example

Example 5

Evaluate $\int_C x\,ds$ for each of the following curves.

(a) $C_1 : y = x^2,\ -1 \le x \le 1$

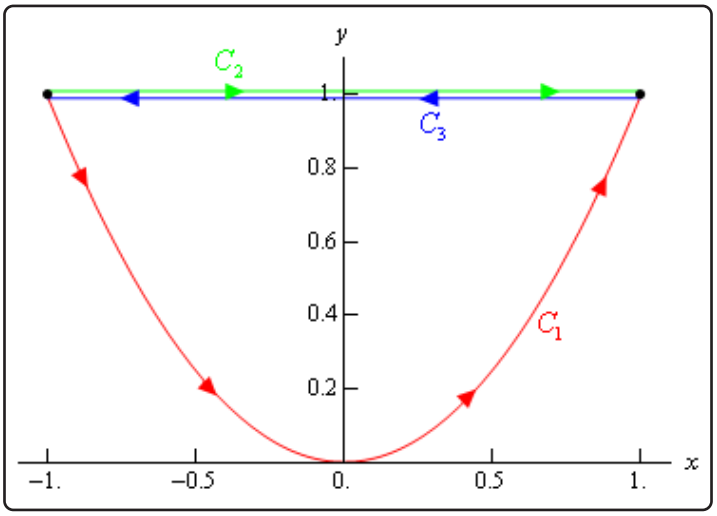
(b) C_2: The line segment from $(-1,1)$ to $(1,1)$.

(c) C_3: The line segment from $(1,1)$ to $(-1,1)$.

Solution

Before working any of these line integrals let's notice that all of these curves are paths that connect the points $(-1,1)$ and $(1,1)$. Also notice that $C_3 = -C_2$ and so by the fact above these two should give the same answer.

Here is a sketch of the three curves and note that the curves illustrating C_2 and C_3 have been separated a little to show that they are separate curves in some way even though they are the same line.



(a) $C_1 : y = x^2,\ -1 \le x \le 1$

Here is a parameterization for this curve.

$$C_1 : x = t,\ y = t^2,\ -1 \le t \le 1$$

Here is the line integral.

$$\int_{C_1} x\,ds = \int_{-1}^{1} t\sqrt{1+4t^2}\,dt = \frac{1}{12}\left(1+4t^2\right)^{\frac{3}{2}}\Big|_{-1}^{1} = 0$$

(b) C_2: The line segment from $(-1,1)$ to $(1,1)$.

There are two parameterizations that we could use here for this curve. The first is to use the formula we used in the previous couple of examples. That parameterization is,

$$\begin{aligned} C_2 : \vec{r}(t) &= (1-t)\langle -1,1\rangle + t\langle 1,1\rangle \\ &= \langle 2t-1,1\rangle \end{aligned}$$

for $0 \le t \le 1$.

Sometimes we have no choice but to use this parameterization. However, in this case there is a second (probably) easier parameterization. The second one uses the fact that we are really just graphing a portion of the line $y = 1$. Using this the parameterization is,

$$C_2 : x = t,\ y = 1,\ \ -1 \le t \le 1$$

This will be a much easier parameterization to use so we will use this. Here is the line integral for this curve.

$$\int_{C_2} x\,ds = \int_{-1}^{1} t\sqrt{1+0}\,dt = \left.\frac{1}{2}t^2\right|_{-1}^{1} = 0$$

Note that this time, unlike the line integral we worked with in Examples 2, 3, and 4 we got the same value for the integral despite the fact that the path is different. This will happen on occasion. We should also not expect this integral to be the same for all paths between these two points. At this point all we know is that for these two paths the line integral will have the same value. It is completely possible that there is another path between these two points that will give a different value for the line integral.

(c) C_3: The line segment from $(1,1)$ to $(-1,1)$.

Now, according to our fact above we really don't need to do anything here since we know that $C_3 = -C_2$. The fact tells us that this line integral should be the same as the second part (*i.e.* zero). However, let's verify that, plus there is a point we need to make here about the parameterization.

Here is the parameterization for this curve.

$$\begin{aligned} C_3 : \vec{r}(t) &= (1-t)\langle 1,1\rangle + t\langle -1,1\rangle \\ &= \langle 1-2t,1\rangle \end{aligned}$$

for $0 \le t \le 1$.

Note that this time we can't use the second parameterization that we used in part (b) since we need to move from right to left as the parameter increases and the second parameterization used in the previous part will move in the opposite direction.

Here is the line integral for this curve.

$$\int_{C_3} x\,ds = \int_0^1 (1-2t)\sqrt{4+0}\,dt = 2\left(t-t^2\right)\Big|_0^1 = 0$$

Sure enough we got the same answer as the second part.

To this point in this section we've only looked at line integrals over a two-dimensional curve. However, there is no reason to restrict ourselves like that. We can do line integrals over three-dimensional curves as well.

Let's suppose that the three-dimensional curve C is given by the parameterization,

$$x = x(t), \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

then the line integral is given by,

Fact

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

Note that often when dealing with three-dimensional space the parameterization will be given as a vector function.

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$$

Notice that we changed up the notation for the parameterization a little. Since we rarely use the function names we simply kept the x, y, and z and added on the (t) part to denote that they may be functions of the parameter.

Also notice that, as with two-dimensional curves, we have,

$$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2} = \left\|\vec{r}\,'(t)\right\|$$

and the line integral can again be written as,

Fact

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\,\left\|\vec{r}\,'(t)\right\|\,dt$$

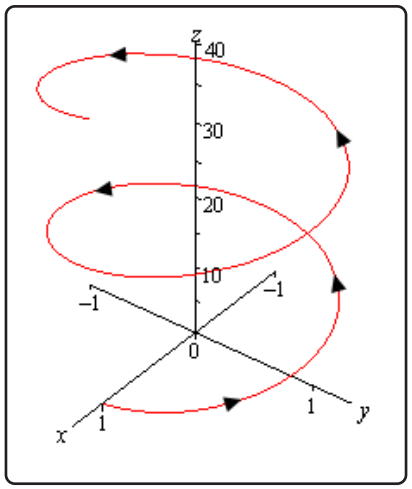
So, outside of the addition of a third parametric equation line integrals in three-dimensional space work the same as those in two-dimensional space. Let's work a quick example.

Example 6

Evaluate $\int_C xyz\,ds$ where C is the helix given by, $\vec{r}(t) = \langle \cos(t), \sin(t), 3t \rangle$, $0 \le t \le 4\pi$.

Solution

Note that we first saw the vector equation for a helix back in the Vector Functions section. Here is a quick sketch of the helix.



Here is the line integral.

$$\begin{aligned}\int_C xyz\,ds &= \int_0^{4\pi} 3t\cos(t)\sin(t)\sqrt{\sin^2(t)+\cos^2(t)+9}\,dt \\ &= \int_0^{4\pi} 3t\left(\frac{1}{2}\sin(2t)\right)\sqrt{1+9}\,dt \\ &= \frac{3\sqrt{10}}{2}\int_0^{4\pi} t\sin(2t)\,dt \\ &= \frac{3\sqrt{10}}{2}\left(\frac{1}{4}\sin(2t)-\frac{t}{2}\cos(2t)\right)\Big|_0^{4\pi} \\ &= -3\sqrt{10}\,\pi\end{aligned}$$

You were able to do that integral right? It required integration by parts.

So, as we can see there really isn't too much difference between two- and three-dimensional line integrals.

16.3 Line Integrals - Part II

In the previous section we looked at line integrals with respect to arc length. In this section we want to look at line integrals with respect to x and/or y.

As with the last section we will start with a two-dimensional curve C with parameterization,

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

Fact

The **line integral of f with respect to x** is,

$$\int_C f(x,y)\,dx = \int_a^b f\big(x(t), y(t)\big)x'(t)\,dt$$

The **line integral of f with respect to y** is,

$$\int_C f(x,y)\,dy = \int_a^b f\big(x(t), y(t)\big)y'(t)\,dt$$

Note that the only notational difference between these two and the line integral with respect to arc length (from the previous section) is the differential. These have a dx or dy while the line integral with respect to arc length has a ds. So, when evaluating line integrals be careful to first note which differential you've got so you don't work the wrong kind of line integral.

These two integral often appear together and so we have the following shorthand notation for these cases.

Fact

$$\int_C P\,dx + Q\,dy = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

Let's take a quick look at an example of this kind of line integral.

Example 1

Evaluate $\int_C \sin(\pi y)\,dy + yx^2\,dx$ where C is the line segment from $(0,2)$ to $(1,4)$.

Solution

Here is the parameterization of the curve.

$$\vec{r}(t) = (1-t)\langle 0,2\rangle + t\langle 1,4\rangle = \langle t, 2+2t\rangle \qquad 0 \le t \le 1$$

The line integral is,

$$\begin{aligned}\int_C \sin(\pi y)\,dy + yx^2\,dx &= \int_C \sin(\pi y)\,dy + \int_C yx^2\,dx \\ &= \int_0^1 \sin\left(\pi(2+2t)\right)(2)\,dt + \int_0^1 (2+2t)(t)^2(1)\,dt \\ &= -\frac{1}{\pi}\cos(2\pi + 2\pi t)\Big|_0^1 + \left(\frac{2}{3}t^3 + \frac{1}{2}t^4\right)\Big|_0^1 \\ &= \frac{7}{6}\end{aligned}$$

In the previous section we saw that changing the direction of the curve for a line integral with respect to arc length doesn't change the value of the integral. Let's see what happens with line integrals with respect to x and/or y.

Example 2

Evaluate $\int_C \sin(\pi y)\,dy + yx^2\,dx$ where C is the line segment from $(1,4)$ to $(0,2)$.

Solution

So, we simply changed the direction of the curve. Here is the new parameterization.

$$\vec{r}(t) = (1-t)\langle 1,4\rangle + t\langle 0,2\rangle = \langle 1-t, 4-2t\rangle \qquad 0 \le t \le 1$$

The line integral in this case is,

$$\begin{aligned}\int_C \sin(\pi y)\,dy + yx^2\,dx &= \int_C \sin(\pi y)\,dy + \int_C yx^2\,dx \\ &= \int_0^1 \sin\left(\pi(4-2t)\right)(-2)\,dt + \int_0^1 (4-2t)(1-t)^2(-1)\,dt \\ &= -\frac{1}{\pi}\cos(4\pi - 2\pi t)\Big|_0^1 - \left(-\frac{1}{2}t^4 + \frac{8}{3}t^3 - 5t^2 + 4t\right)\Big|_0^1 \\ &= -\frac{7}{6}\end{aligned}$$

So, switching the direction of the curve got us a different value or at least the opposite sign of the value from the first example. In fact this will always happen with these kinds of line integrals.

Fact

If C is any curve then,

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \qquad \text{and} \qquad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$$

With the combined form of these two integrals we get,

$$\int_{-C} P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy$$

We can also do these integrals over three-dimensional curves as well. In this case we will pick up a third integral (with respect to z) and the three integrals will be.

Fact

$$\begin{aligned}\int_C f(x,y,z)\,dx &= \int_a^b f\big(x(t),y(t),z(t)\big)x'(t)\,dt \\ \int_C f(x,y,z)\,dy &= \int_a^b f\big(x(t),y(t),z(t)\big)y'(t)\,dt \\ \int_C f(x,y,z)\,dz &= \int_a^b f\big(x(t),y(t),z(t)\big)z'(t)\,dt\end{aligned}$$

where the curve C is parameterized by

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

As with the two-dimensional version these three will often occur together so the shorthand we'll be using here is,

Fact

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C P\,(x,y,z)\,dx + \int_C Q\,(x,y,z)\,dy + \int_C R\,(x,y,z)\,dz$$

Let's work an example.

Example 3

Evaluate $\displaystyle\int_C y\,dx + x\,dy + z\,dz$ where C is given by $x = \cos(t)$, $y = \sin(t)$, $z = t^2$, $0 \le t \le 2\pi$.

Solution

So, we already have the curve parameterized so there really isn't much to do other than evaluate the integral.

$$\begin{aligned}
\int_C y\,dx + x\,dy + z\,dz &= \int_C y\,dx + \int_C x\,dy + \int_C z\,dz \\
&= \int_0^{2\pi} \sin(t)\big(-\sin(t)\big)\,dt + \int_0^{2\pi} \cos(t)\big(\cos(t)\big)\,dt + \int_0^{2\pi} t^2\,(2t)\,dt \\
&= -\int_0^{2\pi} \sin^2(t)\,dt + \int_0^{2\pi} \cos^2(t)\,dt + \int_0^{2\pi} 2t^3\,dt \\
&= -\frac{1}{2}\int_0^{2\pi} 1 - \cos\,(2t)\,dt + \frac{1}{2}\int_0^{2\pi} 1 + \cos\,(2t)\,dt + \int_0^{2\pi} 2t^3\,dt \\
&= \left(-\frac{1}{2}\left(t - \frac{1}{2}\sin\,(2t)\right) + \frac{1}{2}\left(t + \frac{1}{2}\sin\,(2t)\right) + \frac{1}{2}t^4\right)\Bigg|_0^{2\pi} \\
&= 8\pi^4
\end{aligned}$$

16.4 Line Integrals of Vector Fields

In the previous two sections we looked at line integrals of functions. In this section we are going to evaluate line integrals of vector fields. We'll start with the vector field,

$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k}$$

and the three-dimensional, smooth curve given by

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k} \qquad a \le t \le b$$

Fact

The **line integral of $\vec{F}$ along C** is

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}\big(\vec{r}(t)\big)\cdot\vec{r}\,'(t)\;dt$$

Note the notation in the integral on the left side. That really is a dot product of the vector field and the differential really is a vector. Also, $\vec{F}(\vec{r}(t))$ is a shorthand for,

$$\vec{F}(\vec{r}(t)) = \vec{F}\big(x(t), y(t), z(t)\big)$$

We can also write line integrals of vector fields as a line integral with respect to arc length as follows,

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{T}\,ds$$

where $\vec{T}(t)$ is the unit tangent vector and is given by,

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

If we use our knowledge on how to compute line integrals with respect to arc length we can see that this second form is equivalent to the first form given above.

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_C \vec{F}\cdot\vec{T}\,ds\\ &= \int_a^b \vec{F}\big(\vec{r}(t)\big)\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\;\|\vec{r}\,'(t)\|\;dt\\ &= \int_a^b \vec{F}\big(\vec{r}(t)\big)\cdot\vec{r}\,'(t)\;dt\end{aligned}$$

In general, we use the first form to compute these line integral as it is usually much easier to use. Let's take a look at a couple of examples.

Example 1

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = 8x^2yz\,\vec{i} + 5z\,\vec{j} - 4xy\,\vec{k}$ and C is the curve given by $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + t^3\,\vec{k}$, $0 \le t \le 1$.

Solution

Okay, we first need the vector field evaluated along the curve.

$$\vec{F}\left(\vec{r}(t)\right) = 8t^2\left(t^2\right)\left(t^3\right)\vec{i} + 5t^3\,\vec{j} - 4t\left(t^2\right)\vec{k} = 8t^7\,\vec{i} + 5t^3\,\vec{j} - 4t^3\,\vec{k}$$

Next, we need the derivative of the parameterization.

$$\vec{r}\,'(t) = \vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}$$

Finally, let's get the dot product taken care of.

$$\vec{F}\left(\vec{r}(t)\right) \cdot \vec{r}\,'(t) = 8t^7 + 10t^4 - 12t^5$$

The line integral is then,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 8t^7 + 10t^4 - 12t^5\,dt \\ &= \left(t^8 + 2t^5 - 2t^6\right)\Big|_0^1 \\ &= 1\end{aligned}$$

Example 2

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = xz\,\vec{i} - yz\,\vec{k}$ and C is the line segment from $(-1,2,0)$ to $(3,0,1)$.

Solution

We'll first need the parameterization of the line segment. We saw how to get the parameterization of line segments in the first section on line integrals. We've been using the two dimensional version of this over the last couple of sections. Here is the parameterization for

the line.

$$\begin{aligned}\vec{r}(t) &= (1-t)\langle -1,2,0\rangle + t\langle 3,0,1\rangle \\ &= \langle 4t-1, 2-2t, t\rangle, \qquad 0 \le t \le 1\end{aligned}$$

So, let's get the vector field evaluated along the curve.

$$\begin{aligned}\vec{F}\big(\vec{r}(t)\big) &= (4t-1)(t)\,\vec{i} - (2-2t)(t)\,\vec{k} \\ &= \big(4t^2 - t\big)\,\vec{i} - \big(2t - 2t^2\big)\,\vec{k}\end{aligned}$$

Now we need the derivative of the parameterization.

$$\vec{r}\,'(t) = \langle 4, -2, 1\rangle$$

The dot product is then,

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 4\big(4t^2 - t\big) - \big(2t - 2t^2\big) = 18t^2 - 6t$$

The line integral becomes,

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 18t^2 - 6t\,dt = \big(6t^3 - 3t^2\big)\Big|_0^1 = 3$$

Let's close this section out by doing one of these in general to get a nice relationship between line integrals of vector fields and line integrals with respect to x, y, and z.

Given the vector field $\vec{F}(x,y,z) = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ and the curve C parameterized by $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$, $a \le t \le b$ the line integral is,

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_a^b \Big(P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}\Big)\cdot\Big(x'\,\vec{i} + y'\,\vec{j} + z'\,\vec{k}\Big)\,dt \\ &= \int_a^b Px' + Qy' + Rz'\,dt \\ &= \int_a^b Px'\,dt + \int_a^b Qy'\,dt + \int_a^b Rz'\,dt \\ &= \int_C P\,dx + \int_C Q\,dy + \int_C R\,dz \\ &= \int_C P\,dx + Q\,dy + R\,dz\end{aligned}$$

So, we see that,

Fact

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

Note that this gives us another method for evaluating line integrals of vector fields.

This also allows us to say the following about reversing the direction of the path with line integrals of vector fields.

Fact

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

This should make some sense given that we know that this is true for line integrals with respect to x, y, and/or z and that line integrals of vector fields can be defined in terms of line integrals with respect to x, y, and z.

16.5 Fundamental Theorem for Line Integrals

In Calculus I we had the Fundamental Theorem of Calculus that told us how to evaluate definite integrals. This told us,

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

It turns out that there is a version of this for line integrals over certain kinds of vector fields. Here it is.

Theorem

Suppose that C is a smooth curve given by $\vec{r}(t)$, $a \le t \le b$. Also suppose that f is a function whose gradient vector, ∇f, is continuous on C. Then,

$$\int_C \nabla f \cdot d\vec{r} = f\big(\vec{r}(b)\big) - f\big(\vec{r}(a)\big)$$

Note that $\vec{r}(a)$ represents the initial point on C while $\vec{r}(b)$ represents the final point on C. Also, we did not specify the number of variables for the function since it is really immaterial to the theorem. The theorem will hold regardless of the number of variables in the function.

Proof

This is a fairly straight forward proof.

For the purposes of the proof we'll assume that we're working in three dimensions, but it can be done in any dimension.

Let's start by just computing the line integral.

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b \nabla f\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)\; dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt\end{aligned}$$

Now, at this point we can use the Chain Rule to simplify the integrand as follows,

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt \\ &= \int_a^b \frac{d}{dt}\Big[f\left(\vec{r}(t)\right)\Big]\, dt\end{aligned}$$

To finish this off we just need to use the Fundamental Theorem of Calculus for single integrals.

$$\int_C \nabla f \cdot d\vec{r} = f\big(\vec{r}(b)\big) - f\big(\vec{r}(a)\big)$$

Let's take a quick look at an example of using this theorem.

Example 1

Evaluate $\displaystyle\int_C \nabla f \cdot d\vec{r}$ where $f(x,y,z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is *any* path that starts at $\left(1, \frac{1}{2}, 2\right)$ and ends at $(2,1,-1)$.

Solution

First let's notice that we didn't specify the path for getting from the first point to the second point. The reason for this is simple. The theorem above tells us that all we need are the initial and final points on the curve in order to evaluate this kind of line integral.

So, let $\vec{r}(t)$, $a \le t \le b$ be any path that starts at $\left(1, \frac{1}{2}, 2\right)$ and ends at $(2,1,-1)$. Then,

$$\vec{r}(a) = \left\langle 1, \frac{1}{2}, 2 \right\rangle \qquad \vec{r}(b) = \langle 2, 1, -1 \rangle$$

The integral is then,

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= f(2,1,-1) - f\left(1, \frac{1}{2}, 2\right) \\ &= \cos(2\pi) + \sin\pi - 2(1)(-1) - \left(\cos\pi + \sin\left(\frac{\pi}{2}\right) - 1\left(\frac{1}{2}\right)(2)\right) \\ &= 4\end{aligned}$$

Notice that we also didn't need the gradient vector to actually do this line integral. However, for the practice of finding gradient vectors here it is,

$$\nabla f = \langle -\pi \sin(\pi x) - yz, \pi \cos(\pi y) - xz, -xy \rangle$$

The most important idea to get from this example is not how to do the integral as that's pretty simple, all we do is plug the final point and initial point into the function and subtract the two results. The important idea from this example (and hence about the Fundamental Theorem of Calculus) is that, for these kinds of line integrals, we didn't really need to know the path to get the answer. In other

words, we could use any path we want and we'll always get the same results.

In the first section on line integrals (even though we weren't looking at vector fields) we saw that often when we change the path we will change the value of the line integral. We now have a type of line integral for which we know that changing the path will NOT change the value of the line integral.

Let's formalize this idea up a little. Here are some definitions. The first one we've already seen before, but it's been a while and it's important in this section so we'll give it again. The remaining definitions are new.

Definitions

First suppose that $\vec{F}$ is a continuous vector field in some domain D.

1. $\vec{F}$ is a **conservative** vector field if there is a function f such that $\vec{F} = \nabla f$. The function f is called a **potential function** for the vector field. We first saw this definition in the first section of this chapter.

2. $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ is **independent of path** if $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 and C_2 in D with the same initial and final points.

3. A path C is called **closed** if its initial and final points are the same point. For example, a circle is a closed path.

4. A path C is **simple** if it doesn't cross itself. A circle is a simple curve while a figure 8 type curve is not simple.

5. A region D is **open** if it doesn't contain any of its boundary points.

6. A region D is **connected** if we can connect any two points in the region with a path that lies completely in D.

7. A region D is **simply-connected** if it is connected and it contains no holes. We won't need this one until the next section, but it fits in with all the other definitions given here so this was a natural place to put the definition.

With these definitions we can now give some nice facts.

Facts

1. $\int_C \nabla f \cdot d\vec{r}$ is independent of path. This is easy enough to prove since all we need to do is look at the theorem above. The theorem tells us that in order to evaluate this integral all we need are the initial and final points of the curve. This in turn tells us that the line integral must be independent of path.
2. If $\vec{F}$ is a conservative vector field then $\int_C \vec{F} \cdot d\vec{r}$ is independent of path. This fact is also easy enough to prove. If $\vec{F}$ is conservative then it has a potential function, f, and so the line integral becomes $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$. Then using the first fact we know that this line integral must be independent of path.
3. If $\vec{F}$ is a continuous vector field on an open connected region D and if $\int_C \vec{F} \cdot d\vec{r}$ is independent of path (for any path in D) then $\vec{F}$ is a conservative vector field on D.
4. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path then $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C.
5. If $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C then $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

These are some nice facts to remember as we work with line integrals over vector fields. Also notice that 2 & 3 and 4 & 5 are converses of each other.

16.6 Conservative Vector Fields

In the previous section we saw that if we knew that the vector field $\vec{F}$ was conservative then $\int_C \vec{F} \cdot d\vec{r}$ was independent of path. This in turn means that we can easily evaluate this line integral provided we can find a potential function for $\vec{F}$.

In this section we want to look at two questions. First, given a vector field $\vec{F}$ is there any way of determining if it is a conservative vector field? Secondly, if we know that $\vec{F}$ is a conservative vector field how do we go about finding a potential function for the vector field?

The first question is easy to answer at this point if we have a two-dimensional vector field. For higher dimensional vector fields we'll need to wait until the final section in this chapter to answer this question. With that being said let's see how we do it for two-dimensional vector fields.

Theorem

Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open and simply-connected region D. Then if P and Q have continuous first order partial derivatives in D and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

the vector field $\vec{F}$ is conservative.

Let's take a look at a couple of examples.

Example 1

Determine if the following vector fields are conservative or not.

(a) $\vec{F}(x,y) = \left(x^2 - yx\right)\vec{i} + \left(y^2 - xy\right)\vec{j}$

(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Solution

Okay, there really isn't too much to these. All we do is identify P and Q then take a couple of derivatives and compare the results.

(a) $\vec{F}(x,y) = \left(x^2 - yx\right)\vec{i} + \left(y^2 - xy\right)\vec{j}$

In this case here is P and Q and the appropriate partial derivatives.

$$P = x^2 - yx \qquad \frac{\partial P}{\partial y} = -x$$
$$Q = y^2 - xy \qquad \frac{\partial Q}{\partial x} = -y$$

So, since the two partial derivatives are not the same this vector field is NOT conservative.

(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Here is P and Q as well as the appropriate derivatives.

$$P = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} \qquad \frac{\partial P}{\partial y} = 2x^2\mathbf{e}^{xy} + x^2\mathbf{e}^{xy} + x^3y\mathbf{e}^{xy} = 3x^2\mathbf{e}^{xy} + x^3y\mathbf{e}^{xy}$$
$$Q = x^3\mathbf{e}^{xy} + 2y \qquad \frac{\partial Q}{\partial x} = 3x^2\mathbf{e}^{xy} + x^3y\mathbf{e}^{xy}$$

The two partial derivatives are equal and so this is a conservative vector field.

Now that we know how to identify if a two-dimensional vector field is conservative we need to address how to find a potential function for the vector field. This is actually a fairly simple process. First, let's assume that the vector field is conservative and so we know that a potential function, $f(x,y)$ exists. We can then say that,

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = P\vec{i} + Q\vec{j} = \vec{F}$$

Or by setting components equal we have,

Fact

$$\frac{\partial f}{\partial x} = P \qquad \text{and} \qquad \frac{\partial f}{\partial y} = Q$$

By integrating each of these with respect to the appropriate variable we can arrive at the following two equations.

Fact

$$f(x,y) = \int P(x,y)\,dx \qquad \text{or} \qquad f(x,y) = \int Q(x,y)\,dy$$

We saw this kind of integral briefly at the end of the section on iterated integrals in the previous chapter.

It is usually best to see how we use these two facts to find a potential function in an example or two.

Example 2

Determine if the following vector fields are conservative and find a potential function for the vector field if it is conservative.

(a) $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$

(b) $\vec{F}\left(x,y\right) = \left(2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Solution

(a) $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$

Let's first identify P and Q and then check that the vector field is conservative.

$$P = 2x^3y^4 + x \qquad \frac{\partial P}{\partial y} = 8x^3y^3$$
$$Q = 2x^4y^3 + y \qquad \frac{\partial Q}{\partial x} = 8x^3y^3$$

So, the vector field is conservative. Now let's find the potential function. From the first fact above we know that,

$$\frac{\partial f}{\partial x} = 2x^3y^4 + x \qquad \frac{\partial f}{\partial y} = 2x^4y^3 + y$$

From these we can see that

$$f\left(x,y\right) = \int 2x^3y^4 + x\,dx \qquad \text{or} \qquad f\left(x,y\right) = \int 2x^4y^3 + y\,dy$$

We can use either of these to get the process started. Recall that we are going to have to be careful with the "constant of integration" which ever integral we choose to use. For this example let's work with the first integral and so that means that we are asking what function did we differentiate with respect to x to get the integrand. This means that the "constant of integration" is going to have to be a function of y since any function consisting only of y and/or constants will differentiate to zero when taking the partial derivative with respect to x.

Here is the first integral.

$$\begin{aligned} f(x,y) &= \int 2x^3y^4 + x\,dx \\ &= \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + h(y) \end{aligned}$$

where $h(y)$ is the "constant of integration".

We now need to determine $h(y)$. This is easier than it might at first appear to be. To get to this point we've used the fact that we knew P, but we will also need to use the fact that we know Q to complete the problem. Recall that Q is really the derivative of f with respect to y. So, if we differentiate our function with respect to y we know what it should be.

So, let's differentiate f (including the $h(y)$) with respect to y and set it equal to Q since that is what the derivative is supposed to be.

$$\frac{\partial f}{\partial y} = 2x^4y^3 + h'(y) = 2x^4y^3 + y = Q$$

From this we can see that,

$$h'(y) = y$$

Notice that since $h'(y)$ is a function only of y so if there are any x's in the equation at this point we will know that we've made a mistake. At this point finding $h(y)$ is simple.

$$h(y) = \int h'(y)\,dy = \int y\,dy = \frac{1}{2}y^2 + c$$

So, putting this all together we can see that a potential function for the vector field is,

$$f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$$

Note that we can always check our work by verifying that $\nabla f = \vec{F}$. Also note that because the c can be anything there are an infinite number of possible potential functions, although they will only vary by an additive constant.

(b) $\vec{F}(x,y) = \left(2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\right)\vec{i} + \left(x^3\mathbf{e}^{xy} + 2y\right)\vec{j}$

Okay, this one will go a lot faster since we don't need to go through as much explanation. We've already verified that this vector field is conservative in the first set of examples so we won't bother redoing that.

Let's start with the following,

$$\frac{\partial f}{\partial x} = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} \qquad \frac{\partial f}{\partial y} = x^3\mathbf{e}^{xy} + 2y$$

This means that we can do either of the following integrals,

$$f(x,y) = \int 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy}\,dx \qquad \text{or} \qquad f(x,y) = \int x^3\mathbf{e}^{xy} + 2y\,dy$$

While we can do either of these the first integral would be somewhat unpleasant as we would need to do integration by parts on each portion. On the other hand, the second integral is fairly simple since the second term only involves y's and the first term can be done with the substitution $u = xy$. So, from the second integral we get,

$$f(x,y) = x^2\mathbf{e}^{xy} + y^2 + h(x)$$

Notice that this time the "constant of integration" will be a function of x. If we differentiate this with respect to x and set equal to P we get,

$$\frac{\partial f}{\partial x} = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} + h'(x) = 2x\mathbf{e}^{xy} + x^2y\mathbf{e}^{xy} = P$$

So, in this case it looks like,

$$h'(x) = 0 \qquad \Rightarrow \qquad h(x) = c$$

So, in this case the "constant of integration" really was a constant. Sometimes this will happen and sometimes it won't.

Here is the potential function for this vector field.

$$f(x,y) = x^2\mathbf{e}^{xy} + y^2 + c$$

Now, as noted above we don't have a way (yet) of determining if a three-dimensional vector field is conservative or not. However, if we are given that a three-dimensional vector field is conservative finding a potential function is similar to the above process, although the work will be a little more involved.

In this case we will use the fact that,

$$\nabla f = \frac{\partial f}{\partial x}\,\vec{i} + \frac{\partial f}{\partial y}\,\vec{j} + \frac{\partial f}{\partial z}\,\vec{k} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k} = \vec{F}$$

Let's take a quick look at an example.

Example 3

Find a potential function for the vector field,

$$\vec{F} = 2xy^3z^4\,\vec{i} + 3x^2y^2z^4\,\vec{j} + 4x^2y^3z^3\,\vec{k}$$

Solution

Okay, we'll start off with the following equalities.

$$\frac{\partial f}{\partial x} = 2xy^3z^4 \qquad \frac{\partial f}{\partial y} = 3x^2y^2z^4 \qquad \frac{\partial f}{\partial z} = 4x^2y^3z^3$$

To get started we can integrate the first one with respect to x, the second one with respect to y, or the third one with respect to z. Let's integrate the first one with respect to x.

$$f(x,y,z) = \int 2xy^3z^4\,dx = x^2y^3z^4 + g(y,z)$$

Note that this time the "constant of integration" will be a function of both y and z since differentiating anything of that form with respect to x will differentiate to zero.

Now, we can differentiate this with respect to y and set it equal to Q. Doing this gives,

$$\frac{\partial f}{\partial y} = 3x^2y^2z^4 + g_y(y,z) = 3x^2y^2z^4 = Q$$

Of course we'll need to take the partial derivative of the constant of integration since it is a function of two variables. It looks like we've now got the following,

$$g_y(y,z) = 0 \qquad \Rightarrow \qquad g(y,z) = h(z)$$

Since differentiating $g(y,z)$ with respect to y gives zero then $g(y,z)$ could at most be a function of z. This means that we now know the potential function must be in the following form.

$$f(x,y,z) = x^2y^3z^4 + h(z)$$

To finish this out all we need to do is differentiate with respect to z and set the result equal to R.

$$\frac{\partial f}{\partial z} = 4x^2y^3z^3 + h'(z) = 4x^2y^3z^3 = R$$

So,

$$h'(z) = 0 \qquad \Rightarrow \qquad h(z) = c$$

The potential function for this vector field is then,

$$f(x,y,z) = x^2y^3z^4 + c$$

Note that to keep the work to a minimum we used a fairly simple potential function for this example. It might have been possible to guess what the potential function was based simply on the vector field. However, we should be careful to remember that this usually won't be the case and often this process is required.

Also, there were several other paths that we could have taken to find the potential function. Each would have gotten us the same result.

Let's work one more slightly (and only slightly) more complicated example.

Example 4

Find a potential function for the vector field,

$$\vec{F} = \left(2x\cos(y) - 2z^3\right)\vec{i} + \left(3 + 2y\mathbf{e}^z - x^2\sin(y)\right)\vec{j} + \left(y^2\mathbf{e}^z - 6xz^2\right)\vec{k}$$

Solution

Here are the equalities for this vector field.

$$\frac{\partial f}{\partial x} = 2x\cos(y) - 2z^3 \qquad \frac{\partial f}{\partial y} = 3 + 2y\mathbf{e}^z - x^2\sin(y) \qquad \frac{\partial f}{\partial z} = y^2\mathbf{e}^z - 6xz^2$$

For this example let's integrate the third one with respect to z.

$$f(x,y,z) = \int y^2\mathbf{e}^z - 6xz^2\,dz = y^2\mathbf{e}^z - 2xz^3 + g(x,y)$$

The "constant of integration" for this integration will be a function of both x and y.

Now, we can differentiate this with respect to x and set it equal to P. Doing this gives,

$$\frac{\partial f}{\partial x} = -2z^3 + g_x(x,y) = 2x\cos(y) - 2z^3 = P$$

So, it looks like we've now got the following,

$$g_x(x,y) = 2x\cos(y) \qquad \Rightarrow \qquad g(x,y) = x^2\cos(y) + h(y)$$

The potential function for this problem is then,

$$f(x,y,z) = y^2\mathbf{e}^z - 2xz^3 + x^2\cos(y) + h(y)$$

To finish this out all we need to do is differentiate with respect to y and set the result equal to Q.

$$\frac{\partial f}{\partial y} = 2y\mathbf{e}^z - x^2\sin(y) + h'(y) = 3 + 2y\mathbf{e}^z - x^2\sin(y) = Q$$

So,

$$h'(y) = 3 \qquad \Rightarrow \qquad h(y) = 3y + c$$

The potential function for this vector field is then,

$$f(x,y,z) = y^2 \mathbf{e}^{z} - 2xz^3 + x^2 \cos(y) + 3y + c$$

So, a little more complicated than the others and there are again many different paths that we could have taken to get the answer.

We need to work one final example in this section.

Example 5

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \left(2x^3y^4 + x\right)\vec{i} + \left(2x^4y^3 + y\right)\vec{j}$ and C is given by $\vec{r}(t) = (t\cos(\pi t) - 1)\,\vec{i} + \sin\left(\frac{\pi t}{2}\right)\vec{j}$, $0 \le t \le 1$.

Solution

Now, we could use the techniques we discussed when we first looked at line integrals of vector fields however that would be particularly unpleasant solution.

Instead, let's take advantage of the fact that we know from Example 2a above this vector field is conservative and that a potential function for the vector field is,

$$f(x,y) = \frac{1}{2}x^4y^4 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$$

Using this we know that integral must be independent of path and so all we need to do is use the theorem from the previous section to do the evaluation.

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f\left(\vec{r}(1)\right) - f\left(\vec{r}(0)\right)$$

where,

$$\vec{r}(1) = \langle -2, 1\rangle \qquad \vec{r}(0) = \langle -1, 0\rangle$$

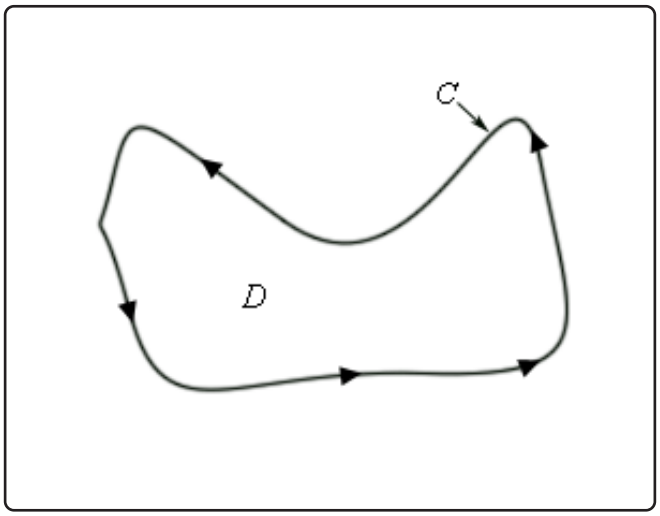
So, the integral is,

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= f(-2,1) - f(-1,0)\\ &= \left(\frac{21}{2} + c\right) - \left(\frac{1}{2} + c\right)\\ &= 10\end{aligned}$$

16.7 Green's Theorem

In this section we are going to investigate the relationship between certain kinds of line integrals (on closed paths) and double integrals.

Let's start off with a simple (recall that this means that it doesn't cross itself) closed curve C and let D be the region enclosed by the curve. Here is a sketch of such a curve and region.



First, notice that because the curve is simple and closed there are no holes in the region D. Also notice that a direction has been put on the curve. We will use the convention here that the curve C has a **positive orientation** if it is traced out in a counter-clockwise direction. Another way to think of a positive orientation (that will cover much more general curves as well see later) is that as we traverse the path following the positive orientation the region D must always be on the left.

Given curves/regions such as this we have the following theorem.

Green's Theorem

Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve. If P and Q have continuous first order partial derivatives on D then,

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Before working some examples there are some alternate notations that we need to acknowledge. When working with a line integral in which the path satisfies the condition of Green's Theorem we will often denote the line integral as,

$$\oint_C P\,dx + Q\,dy \qquad \text{or} \qquad \oint_C P\,dx + Q\,dy$$

Both of these notations do assume that C satisfies the conditions of Green's Theorem so be careful in using them.

Also, sometimes the curve C is not thought of as a separate curve but instead as the boundary of some region D and in these cases you may see C denoted as ∂D.

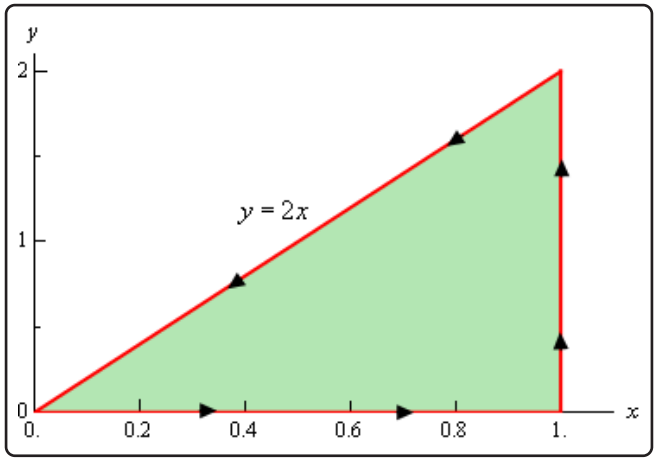
Let's work a couple of examples.

Example 1

Use Green's Theorem to evaluate $\oint_C xy\,dx + x^2y^3\,dy$ where C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.

Solution

Let's first sketch C and D for this case to make sure that the conditions of Green's Theorem are met for C and will need the sketch of D to evaluate the double integral.



So, the curve does satisfy the conditions of Green's Theorem and we can see that the following inequalities will define the region enclosed.

$$0 \le x \le 1 \qquad 0 \le y \le 2x$$

We can identify P and Q from the line integral. Here they are.

$$P = xy \qquad Q = x^2y^3$$

So, using Green's Theorem the line integral becomes,

$$\begin{aligned}\oint_C xy\,dx + x^2y^3\,dy &= \iint_D 2xy^3 - x\,dA \\ &= \int_0^1 \int_0^{2x} 2xy^3 - x\,dy\,dx \\ &= \int_0^1 \left(\frac{1}{2}xy^4 - xy\right)\Big|_0^{2x} dx \\ &= \int_0^1 8x^5 - 2x^2\,dx \\ &= \left(\frac{4}{3}x^6 - \frac{2}{3}x^3\right)\Big|_0^1 \\ &= \frac{2}{3}\end{aligned}$$

Example 2

Evaluate $\oint_C y^3\,dx - x^3\,dy$ where C is the positively oriented circle of radius 2 centered at the origin.

Solution

Okay, a circle will satisfy the conditions of Green's Theorem since it is closed and simple and so there really isn't a reason to sketch it.

Let's first identify P and Q from the line integral.

$$P = y^3 \qquad Q = -x^3$$

Be careful with the minus sign on Q!

Now, using Green's theorem on the line integral gives,

$$\oint_C y^3\,dx - x^3\,dy = \iint_D -3x^2 - 3y^2\,dA$$

where D is a disk of radius 2 centered at the origin.

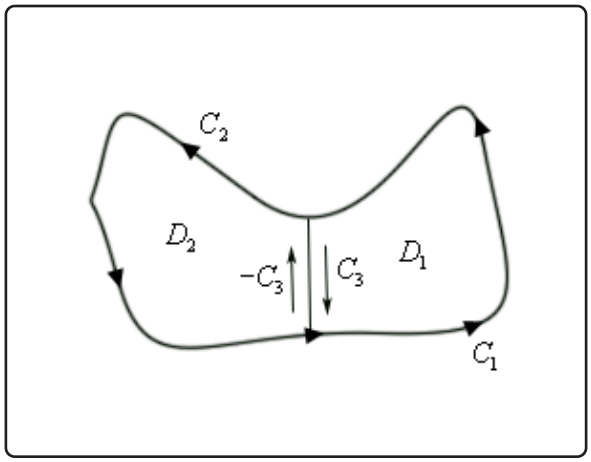
Since D is a disk it seems like the best way to do this integral is to use polar coordinates.

Here is the evaluation of the integral.

$$\begin{aligned}\oint_C y^3\,dx - x^3\,dy &= -3\iint_D \left(x^2+y^2\right)\,dA\\ &= -3\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta\\ &= -3\int_0^{2\pi}\left.\frac{1}{4}r^4\right|_0^2 d\theta\\ &= -3\int_0^{2\pi} 4\,d\theta\\ &= -24\pi\end{aligned}$$

So, Green's theorem, as stated, will not work on regions that have holes in them. However, many regions do have holes in them. So, let's see how we can deal with those kinds of regions.

Let's start with the following region. Even though this region doesn't have any holes in it the arguments that we're going to go through will be similar to those that we'd need for regions with holes in them, except it will be a little easier to deal with and write down.



The region D will be $D_1 \cup D_2$ and recall that the symbol $\cup$ is called the union and means that D consists of both D_1 and D_2. The boundary of D_1 is $C_1 \cup C_3$ while the boundary of D_2 is $C_2 \cup (-C_3)$ and notice that both of these boundaries are positively oriented. As we traverse each boundary the corresponding region is always on the left. Finally, also note that we can think of the whole boundary, C, as,

$$C = (C_1 \cup C_3) \cup (C_2 \cup (-C_3)) = C_1 \cup C_2$$

since both C_3 and $-C_3$ will "cancel" each other out.

Now, let's start with the following double integral and use a basic property of double integrals to

break it up.

$$\iint\limits_D (Q_x - P_y)\, dA = \iint\limits_{D_1 \cup D_2} (Q_x - P_y)\, dA = \iint\limits_{D_1} (Q_x - P_y)\, dA + \iint\limits_{D_2} (Q_x - P_y)\, dA$$

Next, use Green's theorem on each of these and again use the fact that we can break up line integrals into separate line integrals for each portion of the boundary.

$$\begin{aligned}\iint\limits_D (Q_x - P_y)\, dA &= \iint\limits_{D_1} (Q_x - P_y)\, dA + \iint\limits_{D_2} (Q_x - P_y)\, dA \\ &= \oint\limits_{C_1 \cup C_3} P dx + Q dy + \oint\limits_{C_2 \cup (-C_3)} P dx + Q dy \\ &= \oint\limits_{C_1} P dx + Q dy + \oint\limits_{C_3} P dx + Q dy + \oint\limits_{C_2} P dx + Q dy + \oint\limits_{-C_3} P dx + Q dy\end{aligned}$$

Next, we'll use the fact that,

$$\oint\limits_{-C_3} P dx + Q dy = -\oint\limits_{C_3} P dx + Q dy$$

Recall that changing the orientation of a curve with line integrals with respect to x and/or y will simply change the sign on the integral. Using this fact we get,

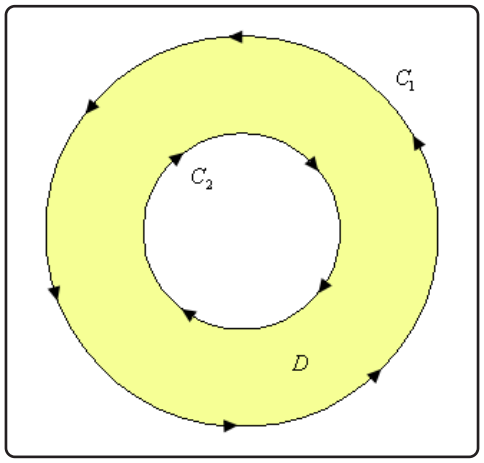
$$\begin{aligned}\iint\limits_D (Q_x - P_y)\, dA &= \oint\limits_{C_1} P dx + Q dy + \oint\limits_{C_3} P dx + Q dy + \oint\limits_{C_2} P dx + Q dy - \oint\limits_{C_3} P dx + Q dy \\ &= \oint\limits_{C_1} P dx + Q dy + \oint\limits_{C_2} P dx + Q dy\end{aligned}$$

Finally, put the line integrals back together and we get,

$$\begin{aligned}\iint\limits_D (Q_x - P_y)\, dA &= \oint\limits_{C_1} P dx + Q dy + \oint\limits_{C_2} P dx + Q dy \\ &= \oint\limits_{C_1 \cup C_2} P dx + Q dy \\ &= \oint\limits_{C} P dx + Q dy\end{aligned}$$

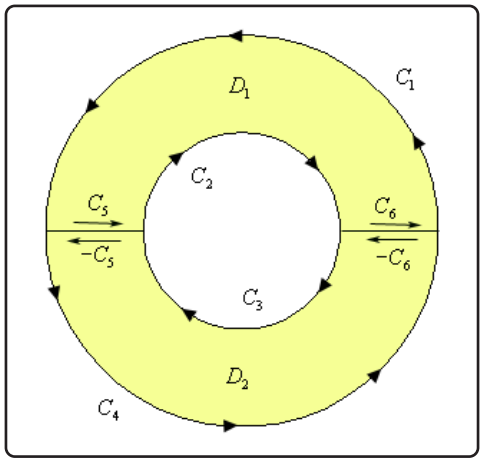
So, what did we learn from this? If you think about it this was just a lot of work and all we got out of it was the result from Green's Theorem which we already knew to be true. What this exercise has shown us is that if we break a region up as we did above then the portion of the line integral on the pieces of the curve that are in the middle of the region (each of which are in the opposite direction) will cancel out. This idea will help us in dealing with regions that have holes in them.

To see this let's look at a ring.



Notice that both of the curves are oriented positively since the region D is on the left side as we traverse the curve in the indicated direction. Note as well that the curve C_2 seems to violate the original definition of positive orientation. We originally said that a curve had a positive orientation if it was traversed in a counter-clockwise direction. However, this was only for regions that do not have holes. For the boundary of the hole this definition won't work and we need to resort to the second definition that we gave above.

Now, since this region has a hole in it we will apparently not be able to use Green's Theorem on any line integral with the curve $C = C_1 \cup C_2$. However, if we cut the disk in half and rename all the various portions of the curves we get the following sketch.



The boundary of the upper portion (D_1)of the disk is $C_1 \cup C_2 \cup C_5 \cup C_6$ and the boundary on the lower portion (D_2)of the disk is $C_3 \cup C_4 \cup (-C_5) \cup (-C_6)$. Also notice that we can use Green's Theorem on each of these new regions since they don't have any holes in them. This means that

we can do the following,

$$\begin{aligned}\iint_D (Q_x - P_y)\,dA &= \iint_{D_1} (Q_x - P_y)\,dA + \iint_{D_2} (Q_x - P_y)\,dA \\ &= \oint_{C_1 \cup C_2 \cup C_5 \cup C_6} P\,dx + Q\,dy + \oint_{C_3 \cup C_4 \cup (-C_5) \cup (-C_6)} P\,dx + Q\,dy\end{aligned}$$

Now, we can break up the line integrals into line integrals on each piece of the boundary. Also recall from the work above that boundaries that have the same curve, but opposite direction will cancel. Doing this gives,

$$\begin{aligned}\iint_D (Q_x - P_y)\,dA &= \iint_{D_1} (Q_x - P_y)\,dA + \iint_{D_2} (Q_x - P_y)\,dA \\ &= \oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy + \oint_{C_3} P\,dx + Q\,dy + \oint_{C_4} P\,dx + Q\,dy\end{aligned}$$

But at this point we can add the line integrals back up as follows,

$$\begin{aligned}\iint_D (Q_x - P_y)\,dA &= \oint_{C_1 \cup C_2 \cup C_3 \cup C_4} P\,dx + Q\,dy \\ &= \oint_C P\,dx + Q\,dy\end{aligned}$$

The end result of all of this is that we could have just used Green's Theorem on the disk from the start even though there is a hole in it. This will be true in general for regions that have holes in them.

Let's take a look at an example.

Example 3

Evaluate $\oint_C y^3\,dx - x^3\,dy$ where C are the two circles of radius 2 and radius 1 centered at the origin with positive orientation.

Solution

Notice that this is the same line integral as we looked at in the second example and only the curve has changed. In this case the region D will now be the region between these two circles and that will only change the limits in the double integral so we'll not put in some of the details here.

Here is the work for this integral.

$$\begin{aligned}\oint_C y^3\,dx - x^3\,dy &= -3\iint_D \left(x^2+y^2\right)\,dA\\ &= -3\int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta\\ &= -3\int_0^{2\pi}\left.\frac{1}{4}r^4\right|_1^2 d\theta\\ &= -3\int_0^{2\pi}\frac{15}{4}\,d\theta\\ &= -\frac{45\pi}{2}\end{aligned}$$

We will close out this section with an interesting application of Green's Theorem. Recall that we can determine the area of a region D with the following double integral.

$$A = \iint_D dA$$

Let's think of this double integral as the result of using Green's Theorem. In other words, let's assume that

$$Q_x - P_y = 1$$

and see if we can get some functions P and Q that will satisfy this.

There are many functions that will satisfy this. Here are some of the more common functions.

$$\begin{array}{lll} P = 0 & P = -y & P = -\dfrac{y}{2}\\ Q = x & Q = 0 & Q = \dfrac{x}{2}\end{array}$$

Then, if we use Green's Theorem in reverse we see that the area of the region D can also be computed by evaluating any of the following line integrals.

Fact

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

where C is the boundary of the region D.

Let's take a quick look at an example of this.

Example 4

Use Green's Theorem to find the area of a disk of radius a.

Solution

We can use either of the integrals above, but the third one is probably the easiest. So,

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

where C is the circle of radius a. So, to do this we'll need a parameterization of C. This is,

$$x = a\cos(t) \qquad y = a\sin(t) \qquad 0 \le t \le 2\pi$$

The area is then,

$$\begin{aligned} A &= \frac{1}{2}\oint_C x\,dy - y\,dx \\ &= \frac{1}{2}\left(\int_0^{2\pi} a\cos(t)\big(a\cos(t)\big)\,dt - \int_0^{2\pi} a\sin(t)\big(-a\sin(t)\big)\,dt\right) \\ &= \frac{1}{2}\int_0^{2\pi} a^2\cos^2(t) + a^2\sin^2(t)\,dt \\ &= \frac{1}{2}\int_0^{2\pi} a^2 dt \\ &= \pi a^2 \end{aligned}$$